

UPPER BOUNDS FOR EIGENVALUES OF CONFORMAL METRICS

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0. Introduction

In this paper we shall study upper bounds for eigenvalues of the Laplacian. In the case of manifolds with boundary we will consider the Neumann eigenvalues μ_k . (We denote Dirichlet eigenvalues by λ_k .) A famous theorem of A. Weyl asserts that given a domain $\Omega \subset \mathbf{R}^n$ with finite volume V , μ_k and λ_k have asymptotic values as $k \rightarrow \infty$, given by $C_n(k/V)^{2/n}$ [13]. Here $C_n \equiv 4\pi^2\omega_n^{-2/n}$, ω_n is the volume of the unit ball in \mathbf{R}^n , and to say that two sequences are asymptotic means that their successive ratios approach 1. It is well known that this asymptotic formula actually holds for any compact Riemannian manifold with boundary. (See e.g. [1].) Of course, the rate at which the eigenvalues become asymptotic to $C_n(k/V)^{2/n}$ depends on the geometry of the domain or manifold one is considering.

G. Polya proved that for certain "tiling domains" in \mathbf{R}^2 the asymptotic formula is actually an estimate below for all the Dirichlet eigenvalues, and an estimate above for all the Neumann eigenvalues [8]. In the same work (and earlier, [7]) he conjectured that these upper and lower bounds should hold for Neumann and Dirichlet eigenvalues on general domains. More precisely, if we define $\mu_1 = 0$ to correspond to the constant function, then Polya conjectured (in the case $n = 2$) that for any finite volume $\Omega \subset \mathbf{R}^n$ we have the estimates

$$(0.1a) \quad \lambda_k \geq C_n \left(\frac{k}{V} \right)^{2/n},$$

$$(0.1b) \quad \mu_k \leq C_n \left(\frac{k-1}{V} \right)^{2/n}.$$

The generality in which such a theorem might be true is not understood. In a survey article S. T. Yau generalizes the conjecture and asks for conditions under which such upper and lower estimates can hold for two-dimensional

(orientable) Riemannian surfaces (with constant C possibly depending on genus) [15]. In this paper we shall only consider upper estimates of the type (0.1b). There is a large literature on lower bounds for eigenvalues which we will not summarize here.

G. Szegő had shown before Polya's result that the first nonconstant Neumann eigenfunction for a simply connected planar domain has eigenvalue bounded above by that of the disk having the same area; i.e., it is bounded by $\rho^2\pi/A$ where $\rho = 1.8412\dots$ is the first positive zero of the appropriate Bessel function [10]. Since $C_2 = 4\pi > \rho^2\pi$, Polya knew that his conjecture held in this case, for μ_2 . J. Hersch used ideas similar to Szegő's to prove that for any Riemannian S^2 there is a sharp upper bound for μ_2 (or by convention $\lambda_1 = \mu_2$), attained with the standard metric, of the form $\lambda_1 \leq 8\pi/A$, where A is again the Riemannian area [3]. (This estimate arises from a stronger sharp inequality for the sum of the reciprocals of the first three eigenvalues.) P. Yang and Yau subsequently used Hersch's idea and a branched-cover argument to conclude analogous (but nonsharp) inequalities for general orientable (M^2, g) of genus g . In particular, λ_1 is bounded above by $8\pi(g+1)/A$ [14]. Li and Yau introduced the concept of conformal volume for a manifold (M, g) and were able to estimate λ_1 above in terms of it and the correct power of actual volume [6]. This gave a new proof of the upper bound estimate for λ_1 quoted above [14]. However the generality of their estimate is limited by the fact that conformal volumes are not easily estimated for general Riemannian manifolds with dimension greater than 2.

Earlier S. Y. Cheng had shown that estimates above do exist for μ_2 (and μ_k) on (M, g) depending linearly on d^{-2} , where d is the diameter of (M, g) , and on a lower bound on the Ricci curvature for M (and on k) [2]. In fact, examples of H. Urakawa show that there does not exist any universal constant C depending only on n so that upper estimates of the form (0.1b) hold for all Riemannian manifolds: he constructs compact manifolds with volume 1 for which $\mu_2 = \lambda_1$ approaches ∞ [12].

In the case of nonnegative Ricci curvature (or more generally Ricci bounded below and diameter bounded above), Li and Yau are able to estimate μ_k from above in terms of volume [5]. For example, if $\text{Ric} \geq 0$ they show the existence of a (large) universal constant (depending only on n) so that

$$(0.2) \quad \mu_k \leq C \left(\frac{k}{V} \right)^{2/n}.$$

It is possible to modify somewhat Weyl's original cube decomposition ideas to show that the upper estimate (0.2) also holds for some large C (depending on n), for arbitrary Euclidean domains. (The author is not aware of any reference to such a theorem in the literature, however.) In the present paper we will adopt a new approach to prove that such upper estimates hold in generality which encompasses more than that of [5] and of Euclidean domains: Namely, we prove estimates for conformal metrics on domains of complete manifolds (M, \mathbf{g}_0) having nonnegative Ricci curvature, or at least having bounds on Ricci · diameter². One consequence is a partial answer to Yau's survey-article question. (See Theorem (0.5) below.)

The method of proof for the main theorem (0.3) involves a decomposition of (M, \mathbf{g}_0) relative to the volume concentration of the conformal metric $\mathbf{g} = \varphi \mathbf{g}_0$. One expects that the ideas in this decomposition could also be useful for other questions.

The results in this paper are the following.

(0.3) **Theorem.** *Let μ_k be the k th Neumann eigenvalue for the Laplacian, with respect to a (finite-volume) conformal metric $\mathbf{g} = \varphi \mathbf{g}_0$ on (a subdomain Ω of) a complete Riemannian manifold (M, \mathbf{g}_0) . Let (M, \mathbf{g}_0) have nonnegative Ricci curvature (assume (M, \mathbf{g}_0) is smooth). Then we have the estimate*

$$\mu_k \leq C \cdot \left(\frac{k}{V} \right)^{2/n},$$

where V is the total volume of (M, \mathbf{g}) (or of (Ω, \mathbf{g})) and C is a (large, nonsharp) constant depending on n , but independent of the particular metric, domain, manifold, or k .

More generally, let (M, \mathbf{g}_0) be complete and satisfy $(\text{Ric})d^2 \geq -a^2$, where Ric is a lower bound for the Ricci curvature on M , d is the diameter of M , and a is a nonnegative constant (if $\text{Ric} = 0$ we take $a = 0$). Then the estimate above holds for μ_k , except that the constant C depends on a as well as n .

As an immediate consequence of the proof of (0.3) we will have

(0.4) **Theorem.** *Let $f: (\widetilde{M}, \mathbf{g}) \rightarrow (M, \mathbf{g}_0)$ be a conformal differentiable map of (topological) degree D , so that every point in M has at most D preimage points in \widetilde{M} and*

$$\langle f_* X, f_* Y \rangle_{\mathbf{g}_0} = \varphi \langle X, Y \rangle_{\mathbf{g}},$$

for a nonnegative function φ on \widetilde{M} . Assume that the \mathbf{g} -volume of $\{x \in \widetilde{M} \text{ such that } f_*: \widetilde{M}_x \rightarrow M_f(x) \text{ satisfies } f_* \equiv 0\}$ is zero. Then, under the same

hypotheses on (M, \mathbf{g}_0) as in (0.3), we have the same estimates for μ_k , except that C is replaced by $C \cdot D^{2/n}$.

Using uniformization theory and the Riemann-Roch theorem from complex analysis, one can quickly apply (0.4) to conformal maps f (arising from meromorphic maps) whose image is S^2 to conclude the estimate:

(0.5) **Theorem.** *Let (M, \mathbf{g}) be a compact (orientable) Riemannian surface of genus g and area A . Then there is a universal constant C so that the k th Neumann eigenvalue μ_k of the Laplacian may be estimated above by*

$$\mu_k \leq C \frac{(g+1)k}{A}.$$

(The idea of using meromorphic functions for this type of estimate originates in [14] and is also used in [6].)

We will prove Theorem (0.3) first. In §1 we list some preliminaries. In §2 we create a natural decomposition of M in terms of \mathbf{g}_0 , which reflects the volume concentration of \mathbf{g} . We prove a key technical lemma (2.5) relating relative volumes for this decomposition. In §3 we apply this lemma to construct sufficiently many Rayleigh-quotient test functions so that our Theorem (0.3) can be proven. In §4 we show how to conclude Theorems (0.4) and (0.5) from the ideas of (0.3). We also include some further remarks.

1. Preliminaries

We specify the configuration for Theorem (0.3): Assume we are given a positive measurable metric $\mathbf{g} = \varphi \mathbf{g}_0$ on a subdomain $\Omega \subset M$. (So also we have the relation $dV = \varphi^{n/2} dV_0$ between the respective volume elements.) Because both sides of the estimate in (0.3) scale by δ^{-1} if the metric is scaled by the positive constant δ , we assume for technical ease that the total volume of (Ω, \mathbf{g}) is 1; it suffices to show the estimate in that case.

(1.1) We will write $m(U)$ for the conformal-metric volume of a measurable set U , and $m_0(U)$ for its volume with respect to \mathbf{g}_0 :

$$m(U) = \int_U \varphi^{n/2} dV_0.$$

For a continuous function h , $\text{supp}(h)$ will be the support of h ; i.e., the set on which it is nonzero.

(1.2) We wish to construct test functions η , and estimate Rayleigh quotients

$$R(\eta) = \frac{\int_{\Omega} |\nabla \eta|^2 dV}{\int_{\Omega} \eta^2 dV}.$$

In the expression for $R(\eta)$, $|\nabla \eta|^2$ is the norm squared of the conformal-metric gradient, $|\nabla \eta|^2 = \varphi^{-1} |D\eta|^2$ (and $|D\eta|^2$ is the norm squared of the gradient with respect to \mathbf{g}_0). It follows from the well-known variational construction of eigenfunctions that if one can find k disjointly-supported functions on Ω , each with Rayleigh quotient bounded above by C , then $\mu_k \leq C$.

We may estimate the Rayleigh quotient by applying Hölder's inequality to the numerator, and using the conformal invariance: $|\nabla \eta|^n dV = |D\eta|^n dV_0$:

$$(1.3) \quad R(\eta) \leq \frac{(\int |D\eta|^n dV_0)^{2/n} m(\text{supp}(D\eta))^{(n-2)/n}}{\int \varphi^{n/2} \eta^2 dV_0}.$$

In (1.2) and (1.3) all integrals were at first assumed to be over the domain Ω . If $\Omega \neq M$ we formally extend φ to be zero on $M \setminus \Omega$. A test function constructed on M gives rise to one for Ω , and its Rayleigh quotient on Ω may be bounded by (1.3), where the integrals are now taken over all of M . Thus in the sequel we assume $\varphi \geq 0$ is defined on all of M , and always use (1.3) to estimate Rayleigh quotients.

(1.4) Our estimates will depend on volume growth of geodesic balls and area growth of geodesic spheres. We collect the relevant constants below. Let a be the nonnegative constant from the hypotheses of (0.3). Pick constant ω_n^a so that

$$(1.4a) \quad \sup_{X,R} R^{-n} m_0(B(X, R)) \leq \omega_n^a.$$

It follows from standard volume comparison theorems, e.g., [4], that ω_n^a may be taken as a^{-n} times the volume of the radius a ball in constant-sectional-curvature space having Ricci curvature -1 . In the case $a = 0$ $\omega_n^a = \omega_n$ suffices.

Pick constant ω_{n-1}^a so that

$$(1.4b) \quad \sup_{x,R} R^{-n+1} \int_{S^{n-1}(x,R)} (\cos \varphi)^+ dA_0 \leq \omega_{n-1}^a.$$

Here $S^{n-1}(x, R)$ is the geodesic sphere of radius R and center x , dA_0 is the induced metric from \mathbf{g}_0 , and φ is the (polar) angle in the tangent

plane M_x between a fixed vector and the tangent vector to the (length-minimizing) geodesic from x to a point on $S^{n-1}(x, R)$. $(\cos \varphi)^+$ is the positive part of $\cos \varphi$. It follows from surface area comparison theorems, e.g., [4], that ω_{n-1}^a may be taken as the value of this normalized integral for a radius a ball in the constant-sectional-curvature space having Ricci curvature -1 . In the case $a = 0$, ω_{n-1}^a can be taken as ω_{n-1} . One can check the following fact (which will be very important in the sequel): $R^{n-1}\omega_{n-1}^a$ majorizes the rate of change (with respect to arclength s) of the (M, \mathbf{g}_0) -volume of a tubular neighborhood of radius R about a geodesic parameterized by arclength s .

Pick (again using volume comparison theorems) constants v_n^a, τ_n^a so that:

(1.4c) Any annulus $B(x, 2R) \setminus B(x, R)$ can be covered by a number of radius $5^{-1}R$ balls which is bounded by v_n^a .

(1.4d) Any $B(x, 50R)$ can be covered by a number of radius R balls which is bounded by τ_n^a .

2. Decomposition of M according to its g -volume

Because we are dealing with a conformal metric we must work on different size scales relative to the background metric \mathbf{g}_0 , in order to account for concentration of the conformal-metric volume. We will construct a family of nested domains in M , related to the local volume density of \mathbf{g} . Our Rayleigh quotient test functions will be supported in these domains or in annular regions between them.

(2.1) Fix $k \in \mathbf{N}$. Let $B(x, R)$ be the open ball of radius R about x with respect to \mathbf{g}_0 . Whenever we speak of distances, it will be with respect to \mathbf{g}_0 . For integer j define

$$(2.1a) \quad S_j = \{B(x, 5^j) \text{ s.t. } m(B) \geq \frac{1}{k}\},$$

$$(2.1b) \quad \Omega_j = \{y \text{ s.t. } y \in B(x, 5^j) \in S_j\} = \bigcup_{B \in S_j} B(x, 5^j).$$

The nesting of the Ω_j (clearly $\Omega_j \subset \Omega_{j+1}$) gives an indication of local volume densities for \mathbf{g} . We construct a related (regularized) family from the S_j, Ω_j as follows:

There is a minimum $j = J$ for which Ω_J and S_J are nonempty. For $j \leq J$ let Θ_j be empty, and for $j \leq J-1$ let β_j be empty. Pick β_J to be a maximal collection of balls $\{B(x_i, 5^J)\} \subset S_J$ which are doubly disjoint;

i.e., the corresponding $B(x_i, 2 \cdot 5^j)$ are disjoint. Let Θ_{j+1} be the $4 \cdot 5^j$ neighborhood of β_j ; i.e., $\Theta_{j+1} = \{\cup B(x_i, 5^{j+1})\}$, where $B(x_i, 5^j) \in \beta_j$.

Inductively define β_j, Θ_{j+1} as follows: Let β_j be a maximal collection of balls in S_j which are doubly disjoint from themselves and from Θ_j . Then define Θ_{j+1} to be the $4 \cdot 5^j$ -neighborhood of the union of Θ_j with the balls in β_j . It is clear (by induction), that $\Theta_j = \cup B(x_i, 5^j)$, where the union is over x_i for which $B(x_i, 5^r) \in \beta_r$, some $r \leq j$.

(2.2) **Lemma.** *We have the inclusions $\Omega_{j-1} \subset \Theta_j \subset \Omega_j$ for all j .*

Proof. $\Theta_j \subset \Omega_j$ is clear from the last sentence of (2.1) above, so we must only show $\Omega_{j-1} \subset \Theta_j$. First prove $\Omega_j \subset \Theta_{j+1}$: Let $y \in \Omega_j$. Then $y \in B(x, 5^j) \in S_j$. Now $B(x, 2 \cdot 5^j)$ must intersect some $B(x_i, 2 \cdot 5^j)$ for which $B(x_i, 5^j) \in \beta_j$, by maximality. From the triangle inequality the distance from y to x_i is less than $5 \cdot 5^j$, so $y \in B(x_i, 5^{j+1}) \subset \Theta_{j+1}$. We prove the general case by induction. Let $y \in \Omega_{j-1}$. Then $y \in B(x, 5^{j-1}) \in S_{j-1}$. $B(x, 2 \cdot 5^{j-1})$ must intersect some $B(z, 2 \cdot 5^{j-1})$ with $B(z, 5^{j-1}) \in \beta_{j-1}$, or else must intersect Θ_{j-1} , by maximality. In the first case y is within $5 \cdot 5^{j-1}$ of z , so is in Θ_j . In the second case y is within $5^{j-1} + 2 \cdot 5^{j-1} + 5^{j-1}$ of some z with $B(z, 5^r) \in \beta_r$, some $r \leq j-1$ (by our characterization of Θ_{j-1} (1.1)). Thus $y \in B(z, 5^j) \subset \Theta_j$. q.e.d.

(2.3) We write $\Theta_j = \cup \Theta_j^i$, where the Θ_j^i are the (connected) components of Θ_j . Similarly, β_j decomposes into a union of balls β_j^i . We will use the collection of sets β_j^i and $\tilde{\Theta}_j^i \equiv \Theta_j^i \setminus (\Theta_{j-1} \cup \beta_{j-1})$ to construct our test functions.

We use the symbol $\#$ to stand for "the number of components." For example, $\#(\Theta_{j-1} \cap \Theta_j^i)$ is the number of components of Θ_{j-1} in Θ_j^i , and $\#(\beta_{j-1} \cap \Theta_j^i)$ is the number of β_{j-1} -balls in Θ_j^i .

Let $M_j^i = m(\tilde{\Theta}_j^i)$, $m_j^i = m(\beta_j^i)$ be conformal metric volumes. Let $E(\tilde{\Theta}_j^i) \equiv E_j^i$, and $E(\beta_j^i) \equiv e_j^i$ be the (Euclidean-)normalized, background-metric volumes of $\tilde{\Theta}_j^i$, $e_j^i : E_j^i \equiv 5^{-nj} m_0(\tilde{\Theta}_j^i)$, $e_j^i \equiv 5^{-nj} m_0(\beta_j^i)$. (By (1.4a) we have $e_j^i \leq \omega_n^a$.)

The reason the \mathbf{g}_0 -volumes are normalized in this fashion is that we will be constructing bump-functions and annularly-supported functions to plug into the Rayleigh-quotient estimate (1.3). Typically they will cut-off in a

piecewise-linear manner, say between some $\partial\Theta_j^i$ and an interior $\partial\Theta_{j-1}^{i'}$ (or $\partial\beta_{j-1}^{i'}$). This will lead to $|D\eta|^n$ of order 5^{-nj} . Hence the first term in the numerator of (1.3) will be estimatable exactly if E_j^i is.

When we use constants in inequalities, they are independent of the manifolds (M, \mathbf{g}) , (M, \mathbf{g}_0) , and the eigenvalue counting index k , but may depend on dimension n and the Ricci-diameter parameter a .

(2.4) **Remark.** The nesting of the domains Θ_j^i and $\beta_j^{i'}$ can be described naturally by a simply connected graph ("tree"), as described below. This tree structure will be important for our later analysis.

For each domain $\tilde{\Theta}_j^i$ and each ball $\beta_j^{i'}$, construct a vertex at "height parameter" j . Two vertices on adjacent levels are connected by an edge (i.e., identified) if and only if the lower-level vertex corresponds either to a $\tilde{\Theta}_{j-1}^{i'}$ or a $\beta_{j-1}^{i'}$ contained in Θ_j^i , where $\tilde{\Theta}_j^i$ is the upper-level vertex. In this case we will say that $\tilde{\Theta}_{j-1}^{i'}$ or $\beta_{j-1}^{i'}$ is a child of $\tilde{\Theta}_j^i$, and $\tilde{\Theta}_j^i$ is the parent of $\tilde{\Theta}_{j-1}^{i'}$ or $\beta_{j-1}^{i'}$.

The "leaves" of this (possibly infinitely high) inverted tree are defined to be vertices connected only to higher-level vertices (children with no offspring), and correspond exactly to particular $\beta_j^{i'}$. For a nonleaf vertex $\tilde{\Theta}_j^i$, define the forking number $f(\tilde{\Theta}_j^i)$ to be the number of children, minus one. (For a leaf vertex, define the forking number to be zero.) Since the Euler characteristic of a (finite) tree is one (the number of vertices minus the number of edges is one for a simply connected graph), and since above a certain "height" our tree has no more forks, we conclude that the number of leaves can be computed as one plus the sum over all nonleaf vertices $\tilde{\Theta}_j^i$ of $f(\tilde{\Theta}_j^i)$. Since forking at vertices will be important in our estimates, we define for a given $\tilde{\Theta}_j^i$ and $s \in \mathbf{Z}$, $s \geq 0$,

$$(2.4a) \quad f_j^i(s) = \sum_{\tilde{\Theta}_{j-s}^{i'} \subset \Theta_j^i} f(\tilde{\Theta}_{j-s}^{i'}).$$

For each leaf $\beta_j^{i'} = B(x, 5^j)$ of the tree we are able to construct a radial test function η which is one on $B(x, 5^j)$, zero outside $B(x, 2 \cdot 5^j)$, and linearly interpolates one and zero between radii 5^j and $2 \cdot 5^j$. By our "doubly-disjoint" construction, these test functions have disjoint supports.

Now we estimate the Rayleigh quotient for such a bump function: Let v_n^a be defined as in (1.4c). the support of $D\eta$ is contained in the annulus $B(x, 2 \cdot 5^j) \setminus B(x, 5^j)$, and this set can be covered with v_n^a balls of radius

5^{j-1} , each of which has conformal metric volume less than $1/k$, since the annulus is in the complement of Ω_{j-1} (2.2). Also, $\int \eta^2 dV$ is at least $1/k$, since $B(x, 5^j) \in \beta_j$. Hence, estimating with (1.3), we have

$$(2.4b) \quad R(\eta) \leq 4(\omega_n^a)^{2/n} (v_n^a)^{(n-2)/n} \cdot k^{2/n}.$$

Thus, if for some fixed fraction $\gamma > 0$ there are more than γk leaves, we will conclude that $\mu_{[\gamma k]+1} \leq C \cdot k^{2/n}$; i.e., $\mu_j \leq C \cdot (j/\gamma)^{2/n}$, for $j = [\gamma k]+1$, the greatest integer function $[\cdot]$, and the constant C from (2.4b). Alternately, it is possible that the number of leaves is $\leq \gamma k$ (for example, if our metric manifold has a long skinny cylindrical region, there may not be many leaves). In this case one wants to construct test functions with support on annular regions obtained from (unions of) the $\tilde{\Theta}_j^i$. The estimates of Lemma (2.5) below will imply that in the case of a small number of leaves, $\#\beta \leq \gamma k$ ($\gamma > 0$ and small to be chosen later), the mass and metric volume of the annular regions can be controlled well enough so that for some $\gamma' > 0$ there are at least $\gamma' k$ annular test functions, each with Rayleigh quotient bounded by a multiple of $k^{2/n}$. This will essentially prove Theorem (0.3).

(2.5) **Basic Lemma.** *Using the notation above we have the estimates for the background and conformal volumes of $\tilde{\Theta}_j^i$:*

$$(2.5a) \quad M_j^i \leq \frac{C_1}{k} \left(1 + 2 \sum_{s=0}^{j-J-1} 5^{-s} f_j^i(s) \right),$$

$$(2.5b) \quad E_j^i \leq \omega_n^a + 2\omega_{n-1}^a \sum_{s=0}^{j-J-1} 5^{-s} f_j^i(s).$$

Proof of (2.5). To prove (2.5a), we note first that by (2.2), $\tilde{\Theta}_j^i$ is in the complement of Ω_{j-2} . Hence any radius 5^{j-2} ball which intersects it has conformal volume bounded above by $1/k$. Therefore it suffices to estimate above the number of such balls required to cover the larger set Θ_j^i , by some constant C_1 times the parenthesized expression in (2.5a). To show (2.5b) we will estimate the background volume of $\tilde{\Theta}_j^i$ above by that of Θ_j^i . In both cases it is useful to consider the digression (2.6) below.

(2.6) **Digression.** Recall (or define) that a (finite connected) metric graph G is a finite collection of "points" or "vertices" $\{z_1, z_2, \dots, z_r\}$, together with a finite set of "edges" and lengths $\{E_\alpha, L_\alpha\}$, where an edge $E_\alpha = E_{ij}^s$ identifies (connects) the points z_i and z_j and has positive

“length” L_α . Out of each edge E_{ij}^s we may construct two oriented edges or “vectors” (or “rays”), V_{ij}^s, V_{ji}^s , rays connecting z_i to z_j and z_j to z_i , respectively, each with length L_{ij}^s .

(2.6a) The natural notion of a path in G is an ordered sequence of vectors V_{ij}^s , where the ending point of one vector in the sequence is the starting point of the next. We assume that any two points in G can be connected by a path. (This is the “connected” assumption.) The length of a path is the sum of the vector lengths. The points in a metric graph inherit a natural metric space structure, by defining the distance between two points as the minimum of all possible lengths of paths connecting them. Distance-realizing paths are called geodesics. Clearly any subpath of a geodesic is also a geodesic.

(2.6b) A subgraph T of G consisting of all the points $\{z_1, z_2, \dots, z_r\}$ and a subset $\{E_\beta\}$ of the edges is called a “maximal tree” for G if T (is connected and) is simply connected, i.e., if T has exactly $r - 1$ edges. Given a graph G one may construct a maximal tree for G as follows: Pick a point, say z_1 . First, take the union of all vectors which are contained in geodesics connecting z_1 to the other points z_2, \dots, z_r . Note, if the vector V_{ij}^t is in this collection, the the vector V_{ji}^t cannot be. Whenever two or more vectors in this union end at a given point z_j , they must both be the last vectors of a geodesic from z_1 to z_j , and so we may delete all but one of them from our collection while still being able to construct geodesics from z_1 to all of z_2, \dots, z_r . Do this deletion process successively to vectors ending at points z_2, \dots, z_r until each $z_j, j \geq 2$, has exactly one vector ending at it. Take T to be $\{z_1, \dots, z_r\}$ together with the edges corresponding with these remaining vectors. Since z_1 can have no vector ending at it, the Euler characteristic of our final graph T is 1, and it is simply connected.

(2.6c) We will call the collection $\{z_1, \dots, z_r\}$ together with $r - 1$ vectors $\{V_\beta\}$ a “directed tree,” DT , of G , with initial point z_1 , if the corresponding collection of edges and points are a maximal tree T of G , and if each V_β is of the form V_{ij} , with $i < j$ (and $j = 2, \dots, r$). We write in this case $i = p(j)$ (meaning i is the predecessor of j). Given a maximal tree T it is easy to construct a directed tree DT as follows: First, repeat the process (2.6b) on T , using the point z_1 as indicated. No vectors can be deleted, since the Euler characteristic is already maximized, so the result will just be to pick “positive” orientations V_{ij} for each edge E_{ij} .

Now we shall renumber z_2, \dots, z_r (and relabel the rays V_{ij}) according

to a natural tracing algorithm: Start at z_1 . Some vector(s) must leave this point. Pick one, and call the endpoint z_2 . Renumber the just traversed ray as V_{12} . Proceed inductively as follows. Assume we have redefined points z_1, z_2, \dots, z_j , and rays $V_{p(i),i}$ for $2 \leq i \leq j$. If z_j has rays leaving it, pick one, call its endpoint z_{j+1} , and $V_{j,j+1}$ is the new ray. (Since each point $z_j, j \geq 2$, is hit by exactly one ray, then z_{j+1} cannot be a point which has already been renamed, assuming by induction that z_j was not.) If z_j has no rays leaving it, then back up to the most recent (renamed) z_i (i.e. $i < j$ with i largest possible), which has so-far untraversed rays leaving it. Pick one of these rays, call its endpoint z_{j+1} , and $V_{i,j+1}$ is the new ray. This inductive process ends after a finite number of steps. The points which are traversed in the process are a connected subset of T , so are all of $\{z_1, \dots, z_r\}$, and we have constructed a directed tree DT .

(2.6d) **Lemma.** *Let $\{z_1, \dots, z_r\} \subset (M^n, \mathbf{g}_0)$ and $\{V_{p(i),i}\}$ ($2 \leq i \leq r$) be a collection of vectors $V_{p(i),i}$ with lengths ρ_i , so that we have a directed tree in the sense of (2.6c). Assume that the lengths ρ_i are actually the distances on (M, \mathbf{g}_0) between $z_{p(i)}$ and z_i . Define $l = \sum \rho_i$ to be the "total length." Let $R > 0$. Then $\{z_1, \dots, z_r\}$ can be covered by $[L/R] + 1$ (or fewer) closed balls of radius R (where $[\cdot]$ is the greatest integer function).*

Proof. To prove the lemma we first construct distance-realizing geodesics γ_i from $z_{p(i)}$ to z_i . Let the cycle Γ be the sum $\gamma_1 + \gamma_2 + \dots + \gamma_r$, parameterized by arclength. (So Γ has length equal to L .) For points $P \in \Gamma$ define the "farthest distance function" $\varphi(P)$, as follows: From P there are a finite number of positively oriented (2.6c) (continuous) paths on Γ (leading to terminal points of the tree). Let $\varphi(P)$ be the maximum length of such a curve.

If $L \leq R$ (or more generally, if $\varphi \leq R$ on all of Γ), then the closed ball $\overline{B}(z_1, R)$ covers $\{z_1, \dots, z_r\}$ and (2.6d) holds. If $\varphi \geq R$ somewhere, pick $P \in \Gamma$ with $\varphi(P) = R$. Then $\overline{B}(P, R)$ covers all of the positively oriented paths emanating from P , and these paths have total length at least R . Therefore, if we construct a new directed tree by deleting the union of these paths (but keeping P), we have decreased the total length by at least R . Repeat the process above on the reduced tree, and after at most $[L/R]$ steps the total remaining length will be less than R . This last remaining piece can be covered by $\overline{B}(z_1, R)$, and the lemma is proven.

(2.6e) **Lemma.** *Let $\{z_1, \dots, z_r\} \subset (M^n, \mathbf{g}_0)$ and $\{V_{p(i),i}\}$ ($2 \leq i \leq r$) be a collection of vectors $V_{p(i),i}$ with lengths ρ_i , so that we have a directed tree in the sense of (2.6c). Assume that the lengths ρ_i are actually*

the distances on (M, \mathbf{g}_0) between $z_{p(i)}$ and z_i . Let $N_R(z_1, z_2, \dots, z_r)$ be the set of points within (\mathbf{g}_0) -distance R of the points $\{z_1, \dots, z_r\}$. Then $m_0(N_R(z_1, z_2, \dots, z_r))$ is bounded above by

$$m_0(N_R(z_1, z_2, \dots, z_r)) \leq \omega_n^a R^n + \omega_{n-1}^a R^{n-1} \sum_{i=2}^r \rho_i.$$

Proof. Note first the geometric meaning of the inequality's right-hand side: If, in \mathbf{R}^n , points z_i are placed on the x^1 axis, with x^1 -coordinates $0, \rho_1, \rho_1 + \rho_2, \dots, \sum \rho_i$, then the right-hand side is exactly the volume of the R -neighborhood of the line segment from z_1 to z_r .

Note also that a cruder version of this estimate is obtainable from (2.6d). Namely, if we take the (open) balls of radius $2R$, with centers determined from the covering process (2.6c), then they cover $N_R(z_1, z_2, \dots, z_r)$. From (2.6c) we conclude that

$$m_0(N_R(z_1, z_2, \dots, z_r)) \leq \omega_n^a (2R)^n (1 + L/R),$$

where L is again $\sum \rho_i$. This estimate would actually suffice to prove our eigenvalue estimates, albeit with even more astronomical constants. We prove the estimate (2.6e) because it is sharp, and easy.

To prove the lemma we mimic the proof of (2.6d): Let $\Gamma = \gamma_1 + \gamma_2 + \dots + \gamma_r$ be constructed and parameterized as in that lemma. For arclength parameter s , let $m_0(s)$ be the \mathbf{g}_0 -volume of the R -neighborhood of $\{\Gamma(t), 0 \leq t \leq s\}$. Then $m_0(s)$ is a Lipschitz function, because our tree is directed in the sense of (2.6c). (That is, every time we jump locations on our cycle, we start from a location at which the R -ball is already in our neighborhood.) $m_0(0)$ is bounded by $\omega_n^a R^n$, by (1.4a). Also, by the discussion in (1.4b), we have

$$(2.6f) \quad \frac{d}{ds} m_0(s) \leq \omega_{n-1}^a R^{n-1}.$$

Upon integrating, we have (2.6e). q.e.d.

(2.7a) We return now to the proof of estimates (2.5a), (2.5b). We wish to construct an optimal directed tree for the points $\{z \in \Theta_j^i \text{ s.t. } B(z, 5^r) \in \beta_r, \text{ some } r \leq j-1\}$, and then apply the estimates (2.6d), (2.6e). We construct our tree via an inductive process on decreasing r .

Consider first the subset of z 's with $B(z, 5^{j-1}) \in \beta_{j-1}$, say z_1, \dots, z_{r_1} . If this set is empty proceed to the induction step. Otherwise, we know by construction (each Θ_j^i is connected) that each of these z_i 's is either within $2 \cdot 5^j$ of another such $z_{i'}$, or within $2 \cdot 5^j$ of a z in our set having $B(z, 5^r) \in \beta_r, r < j-1$. Construct a formal metric graph (2.6) having

vertices for each z_1, \dots, z_{r_1} , as well as for each set $\Theta_{j-1}^{i'} \subset \Theta_j^i$. Add edges between "points" as follows: if the two points are z_r 's, $1 \leq r \leq r_1$, then add an edge (with "length" $2 \cdot 5^j$) if the distance between them is less than $2 \cdot 5^j$. If one vertex is a z_r , $1 \leq r \leq r_1$, and the other is an $\Theta_{j-1}^{i'}$, then add an edge (with "length" $2 \cdot 5^j$) if there is a β -point $z \in \Theta_{j-1}^{i'}$ with $|z_r - z| < 2 \cdot 5^j$. If the vertices are $\Theta_{j-1}^{i'}$ and $\Theta_{j-1}^{i''}$ then add an edge (with "length" $2 \cdot 5^j$) if there are β -points $z_{i'}$ and $z_{i''}$ in the respective sets, with $|z_{i'} - z_{i''}| < 2 \cdot 5^j$.

We have now constructed a connected metric graph, and using (2.6c) we pick a maximal tree T_j^i for it. The number of edges in T_j^i is exactly $f(\tilde{\theta}_j^i) = f_j^i(0)$ (2.4a), and each has length $2 \cdot 5^j$. We now repeat the process inductively on all the $\Theta_{j-s}^{i'}$, $s = 1, 2, \dots$. At each step we construct a lower order maximal tree $T_{j-s}^{i'}$. The total number of edges at level $j-s$ is exactly $f_j^i(s)$, and they each have length $2 \cdot 5^{j-s}$. Now glue the maximal trees together in the natural way to get a tree for our entire set $\{z \in \Theta_j^i \text{ s.t. } B(z, 5^r) \in \beta_r, \text{ some } r \leq j-1\}$: For example, in the first stage we glue a maximal tree $T_{j-1}^{i'}$ to T_j^i as follows. Let DT_j be the directed tree made out of T_j^i , with initial point z_1 . Then there is a unique ray pointing to $\Theta_{j-1}^{i'}$. Replace this with a ray pointing to a $(j-2)$ subset of $\Theta_{j-1}^{i'}$ containing a z -point which was distance less than $2 \cdot 5^j$ from the ray's origin. Construct the directed tree $DT_{j-1}^{i'}$ with this initial point. If there is a j -level ray pointing out of $\Theta_{j-1}^{i'}$, replace it with one pointing out of an appropriate $(j-2)$ level subset.

At the end of our inductive process we have constructed a maximal tree for our entire set, and by applying (2.6c) again if necessary we may assume we have a directed tree DT (we suppress $i-j$ dependence). Letting ρ_i be the actual (M^n, g_0) -distance from $z_{p(i),i}$, we have the estimate for $L = \sum \rho_i$:

$$(2.7b) \quad \sum \rho_i \leq \sum_{s=0}^{j-J-1} 2 \cdot 5^{j-s} f_j^i(s).$$

By our preliminary remarks, to prove (2.5a) it suffices to estimate the number of radius 5^{j-2} balls required to cover Θ_j^i . But (2.6d), with $R = 5^j$, implies that Θ_j^i can be covered by $[L/R] + 1$ radius $2 \cdot 5^j$ balls (take

radius $2 \cdot 5^j$ balls centered at the covering balls' centers). Recalling the definition of τ_n^a (1.4d) we let $C_1 = \tau_n^a$, and the number of 5^{j-2} balls required to cover Θ_j^i is bounded above by $C_1(1 + L/R)$. Substituting in our estimate (2.7b) for L and using $R = 5^j$ yield (2.5a).

Applying Lemma (2.6e) with $R = 5^j$, and using the estimate (2.7b), we immediately get

$$m_0(\Theta_j^i) \leq \omega_n^a 5^{nj} + \omega_{n-1}^a 5^{(n-1)j} \left(\sum_{s=0}^{j-J-1} 2 \cdot 5^{j-s} f_j^i(s) \right).$$

Upon dividing by 5^{nj} , this gives (2.5b).

3. Annular test functions

We use the estimates (2.5a), (2.5b) to show that if the number of leaves on our graph (2.4) is small, then we can construct enough annular test functions to conclude Theorem (0.3). To this end we define $0 < \alpha \leq 1$ by

$$(3.1a) \quad \#\beta \equiv \alpha k,$$

so α is a fraction measuring how many leaves our graph has relative to our subdivision parameter k .

In order for our strategy to succeed, for small α we cannot have too many vertices on our graph with large mass M_j^i (for that will decrease the estimatable number of vertices and annular test functions, as well as cause trouble in the second numerator term of the Rayleigh quotient estimate (1.3)), nor can we have too many vertices with large E_j^i (for that will cause trouble in the first numerator term of (1.3)). We are led therefore to the following two estimates:

$$(3.1b) \quad \sum_{E_j^i \geq 2\omega_n^a} E_j^i \leq 5\omega_{n-1}^a \alpha k \equiv C_2 \alpha k,$$

$$(3.1c) \quad \sum_{M_j^i \geq 2C_1/k} M_j^i \leq 5C_1 \alpha \equiv C_3 \alpha.$$

(And for later reference, define $M_1 = 2C_1$.) These results are immediate consequences of (2.5b) and (2.5a): For example, the proof of (3.1b) is:

$$\begin{aligned} \sum_{E_j^i \geq 2\omega_n^a} E_j^i &\leq 2 \sum_{E_j^i \geq 2\omega_n^a} (E_j^i - \omega_n^a) \leq 4\omega_{n-1}^a \sum_{\tilde{\Theta}_j^i} \sum_{s=0}^{j-J-1} 5^{-s} f_j^i(s) \\ &\leq 4\omega_{n-1}^a \left(\frac{1}{1-1/5} \right) \alpha k = 5\omega_{n-1}^a \alpha k. \end{aligned}$$

In this calculation we applied (2.5b) for the second inequality, and summed over *all* nonleaf vertices $\tilde{\Theta}_j^i$. For the third inequality we interchanged the order of summation and summed the appropriate geometric series. The proof of (3.1c) is formally the same, using (2.5a) in place of (2.5b).

(3.2) Now we partition M^n by taking connected unions of the various sets $\tilde{\Theta}_j^i, \beta_j^i$, into larger sets having conformal-metric volume at least $1/k$. We require this partition to be “maximal” in the sense that none of the unions can be subdivided into smaller admissible pieces. (Of course each β_j^i has conformal-metric volume $\geq 1/k$, so will be in a distinct set of this maximal partitioning, but successive $\tilde{\Theta}_j^i$ ’s may have to be unioned with themselves or elements of β in order to attain mass at least $1/k$.) Call this partitioning $P = \cup P_j$. P has a tree structure; it is obtainable from that of the $\tilde{\Theta}_j^i, \beta_j^i$ sets (1.4) by identifying vertices which belong to the same P_j .

We may speak of the parent $p(P_j)$ of P_j and of any possible children $c(P_j)$, in analogy to the discussion in (2.4). (These will be elements of P .) In $p(P_j)$, the nearest $\tilde{\Theta}_r^s$ to P_j , not in P_j , (the Θ -parent) will be denoted by $\tilde{p}(P_j)$. Similarly we will speak of the $\tilde{\Theta}_r^s$ or β_r^s -children of P_j , denoted by $\tilde{c}(P_j)$. (They are the union of all the children of the constituents of P_j .)

For nonleaf P_j we write $f(P_j)$ for the number of forks, i.e. $f(P_j) = \#c(P_j) - 1$. Each leaf of P will contain a (unique) β_j^i , but because leaves β_j^i of our original graph may be absorbed into nonleaf elements of P , our new graph may have fewer leaves than the original. Thus the sum of $f(P_j)$ is bounded above by $(\#\beta) - 1 = \alpha k - 1$.

We wish to construct test functions η which are $\equiv 1$ on a given P_j , $\equiv 0$ on nonparent and nonchild $\tilde{\Theta}_r^s, \beta_r^s$, and linear functions of the (M, \mathbf{g}_0) -distance to P_j in between. In estimating how many of these we may construct, and in estimating the Rayleigh quotient of each (1.3), it is helpful to know $f(P_j) = 0$ (note this allows for the case that P_j is a leaf (2.4)), $f(\tilde{p}(P_j)) = 0$, and to know bounds for $m(\tilde{p}(P_j)), E(\tilde{p}(P_j)), m(\tilde{c}(P_j))$, and $E(\tilde{c}(P_j))$ (2.3). Accordingly, we define P_j to be “good” if (recall $M_1 \equiv 2C_1$)

$$\begin{aligned}
 (3.2a) \quad & f(P_j) = 0, & f(\tilde{p}(P_j)) = 0, \\
 & E(\tilde{p}(P_j)) \leq 2\omega_n^a, & E(\tilde{c}(P_j)) \leq 2\omega_n^a, \\
 & m(\tilde{p}(P_j)) \leq M_1/k, & m(\tilde{c}(P_j)) \leq M_1/k.
 \end{aligned}$$

If P_j is good define η as above. More precisely, if $\tilde{p}(P_j) = \tilde{\Theta}_r^s$, let $\eta(d) = (1 - 2^{-1} \cdot 5^{-r+1}d)^+$ there, and if $\tilde{c}(P_j) = \tilde{\Theta}_r^{s'}$ exists, let $\eta(d) = (1 - 2^{-1} \cdot 5^{-r'+1}d)^+$ there. (Here d is the function “(\mathbf{g}_0) distance to P_j .”) Estimating the Rayleigh quotient of η from (1.3) yields

$$(3.2b) \quad R(\eta) \leq \left(\frac{25}{2}\right) (2\omega_n^a)^{2/n} M_1^{(n-2)/n} \cdot k^{2/n}.$$

(3.3) The next sequence of inequalities will let us find an estimatable number of good P_j .

We have immediately from the relationship between forks and leaves that

$$(3.3a) \quad \begin{aligned} \#\{P_j \text{ s.t. } f(P_j) \geq 1\} &\leq \alpha k, \\ \#\{P_j \text{ s.t. } f(\tilde{p}(P_j)) \geq 1\} &\leq 2\alpha k. \end{aligned}$$

This last inequality has the factor of 2 because a given set $\tilde{\Theta}_r^s$ may be parent to several P_j , but the number is bounded by $f(\tilde{\Theta}_r^s) + 1 \leq 2f(\tilde{\Theta}_r^s)$, if $f(\tilde{\Theta}_r^s) \geq 1$.

Estimates (3.1b) and (3.1c) give

$$(3.3b) \quad \#\{P_j \text{ s.t. } f(\tilde{p}(P_j)) = 0 \text{ and } E(\tilde{p}(P_j)) \geq 2\omega_n^a\} \leq \frac{C_2}{2\omega_n^a} \alpha k,$$

$$\#\{P_j \text{ s.t. } f(P_j) = 0 \text{ and } E(\tilde{c}(P_j)) \geq 2\omega_n^a\} \leq \frac{C_2}{2\omega_n^a} \alpha k;$$

$$(3.3c) \quad \#\{P_j \text{ s.t. } f(\tilde{p}(P_j)) = 0 \text{ and } m(\tilde{p}(P_j)) \geq M_1/k\} \leq \frac{C_3}{M_1} \alpha k,$$

$$\#\{P_j \text{ s.t. } f(P_j) = 0 \text{ and } m(\tilde{c}(P_j)) \geq M_1/k\} \leq \frac{C_3}{M_1} \alpha k.$$

For a given P_j , let $h(p_j)$ (“heavy”) be the particular $\tilde{\Theta}_r^s$ or β_r^s in P_j with the largest conformal-metric volume. Each connected component of $P_j \setminus h(P_j)$ has mass less than $1/k$, by maximality. Therefore (counting the component in the parent direction and the possible components in the child directions of $h(P_j)$), we have

$$(3.3d) \quad m(P_j) < m(h(P_j)) + \frac{2}{k} + \frac{f(P_j)}{k}.$$

Let

$$(3.3e) \quad M_2 \equiv M_1 + 2.$$

If $m(P_j) \geq M_2/k$, then (3.3d) implies that either $m(h(P_j)) \geq M_1/k$ or $m(h(P_j)) < M_1/k$ and $f(P_j) \geq 1$. Therefore in the case $m(P_j) \geq M_2/k$, we absorb the middle term of the right-hand side of (3.3d) and get one of the corresponding estimates

$$m(P_j) \leq \left(1 + \frac{2}{M_1}\right) m(h(P_j)) + \frac{f(P_j)}{k}, \quad \text{if } m(h(P_j)) \geq \frac{M_1}{k},$$

$$m(P_j) \leq \frac{M_1 + 3f(P_j)}{k}, \quad \text{if } f(P_j) \geq 1 \text{ and } m(h(P_j)) < \frac{M_1}{k}.$$

Summing these inequalities over $m(P_j) \geq M_2/k$, and applying (3.1c), (3.3a) we get

$$(3.3f) \quad \sum_{m(P_j) \geq M_2/k} m(P_j) \leq \left(1 + \frac{2}{M_1}\right) C_3 \alpha + (M_1 + 4) \alpha \equiv C_4 \alpha.$$

Since the volume of (M, \mathbf{g}) was normalized to be one, we know that $\sum m(P_j) = 1$. Therefore (3.3f) yields

$$(3.3g) \quad \#\{P_j\} \geq (1 - C_4 \alpha) \frac{k}{M_2}.$$

Combining (3.3a), (3.3b), (3.3c), and (3.3g) we see that

$$(3.3h) \quad \#\{P_j \text{ s.t. } P_j \text{ good}\} \geq \frac{k}{M_2} - \left(\frac{C_4}{M_2} + 3 + \frac{C_2}{\omega_n^a} + \frac{2C_3}{M_1}\right) \alpha k$$

$$\equiv (\varepsilon_1 - C_5 \alpha) k.$$

(3.4) We may now conclude Theorem (0.3) as follows: First, let C_6 be the larger of the two Rayleigh-quotient bounds from (2.4b) and (3.2b):

$$(3.4a) \quad C_6 = \max\{4(\omega_n^a)^{2/n} (v_n^a)^{(n-2)/n}, \left(\frac{25}{2}\right) (2\omega_n^a)^{2/n} M_1^{(n-2)/n}\}.$$

Let γ be given by

$$(3.4b) \quad \gamma = \frac{\varepsilon_1}{3 + C_5}.$$

Case I: the number of leaves αk satisfies $\alpha > \gamma$. In this case (2.4) implies we may obtain $[\gamma k] + 1$ independent bump-function Rayleigh quotients, each bounded by $C_6 k^{2/n}$.

Case II: the number of leaves αk satisfies $\alpha \leq \gamma$. In this case (3.3h) implies that the number of good P_j is at least $3\gamma k$. Since an annular test function constructed from a good P_j has support on at most two other

good $P_{j'}$, we may construct at least $[\gamma k]$ annular test functions in this case, with Rayleigh quotients bounded by $C_6 k^{2/n}$.

Therefore, for any $k \in \mathbf{Z}^+$, we conclude

$$(3.4c) \quad \mu_{[\gamma k]} \leq C_6 \cdot k^{2/n}.$$

For $j = 1, 2, \dots$ pick $k = [2j/\gamma]$. Then $\gamma k \geq \gamma(2j/\gamma - 1) \geq 2j - \gamma \geq j$, so (3.4c) yields

$$(3.4d) \quad \mu_j \leq C_6 \left(\frac{2j}{\gamma} \right)^{2/n} \equiv C_7 j^{2/n}.$$

Rescaling our conformal metric to arbitrary volume yields Theorem (0.3), with $C = C_7$.

4. Generalizations and related questions

(4.1) We show how Theorem (0.4) follows from the proof of Theorem (0.3). Let $f: (\tilde{M}, \mathbf{g}) \rightarrow (M, \mathbf{g}_0)$ be a conformal differentiable map as in the statement of (0.4). We assume (M, \mathbf{g}_0) is as in (0.3), and we will decompose it as previously, except that for a subset $U \subset M$ we now define $m(U)$ to be the \mathbf{g} -volume of $f^{-1}(U)$ (contrast with (1.1)). We construct test functions on \tilde{M} by pulling back ones on M ; i.e., we use functions of the form $\eta \circ f$.

We compute the relations between corresponding function gradients and volume elements on \tilde{M} and M as follows. At points $x \in \tilde{M}$ where f is a local diffeomorphism we pull back g_0 to \tilde{M}_x , by

$$f^* g_0(X, Y) \equiv \langle f_* X, f_* Y \rangle_{f(x)} \equiv \varphi(x) \langle X, Y \rangle_x.$$

So, locally (on \tilde{M}), (\tilde{M}, \mathbf{g}) is conformal to an isometric copy of (M, \mathbf{g}_0) , with conformal factor φ^{-1} . As in §1, and with φ^{-1} replacing φ , we see that

$$(4.1a) \quad |\nabla(\eta \circ f)|^2 = \varphi |D\eta|^2, \quad dV_g = \varphi^{-n/2} dV_{g_0}.$$

Using (4.1a) we may estimate the Rayleigh quotient of $\eta \circ f$:

$$\begin{aligned} R(\eta \circ f) &\leq \frac{\left(\int_M |\nabla(\eta \circ f)|^n dV_g \right)^{2/n} m(\text{supp}(D\eta))^{(n-2)/n}}{\int_{\tilde{M}} \eta^2 dV_g} \\ &\leq \frac{D^{2/n} \left(\int_M |D\eta|^n dV_{g_0} \right)^{2/n} m(\text{supp}(D\eta))^{(n-2)/n}}{m(\{\eta^2 \geq 1\})}. \end{aligned}$$

It follows from (4.1b) that any Rayleigh quotients for $\eta \circ f$ which we construct using §2 or §3 techniques can be estimated above by $D^{2/n}$ times the old estimates (2.4b), (3.2b).

(4.1c) The additional condition imposed on (0.4), namely that the g -volume of $\{x \in \widetilde{M} \text{ s.t. } f_*: \widetilde{M}_x \rightarrow M_{f(x)} \text{ satisfies } f_* \equiv 0\}$ be zero, guarantees that the definition of $\{\Theta_j, \beta_j\}$ (2.1)–(2.4) can be carried out: Under this condition it follows that for any $1/k$ with $k \in \mathbf{N}$, there is an $N = N(k)$ so that no ball of radius 5^{-N} (or smaller) can contain mass as much as $1/k$. Indeed, the g -volume of $\Omega_\varepsilon \equiv \{x \in \widetilde{M} \text{ s.t. } \varphi(x) \leq \varepsilon\}$ must approach 0 as $\varepsilon \rightarrow 0$. Therefore, picking ε so that the g -volume of Ω_ε is less than $1/2k$, it follows that for $B(y, r) \subset (M, g_0)$

$$m(B(y, r)) \leq 1/2k + D\varepsilon^{-n/2}m_0(B(y, r)).$$

Hence for r sufficiently small, $m(B(y, r)) < 1/k$, independently of y (depending on the constant a (0.3)).

The calculation (4.1b) and the remark (4.1c) are enough to guarantee that the decomposition process of §§2, 3 can be repeated to prove (0.4).

(4.2) Theorem (0.4) implies (0.5) by the following remarks: The Riemann-Roch theorem guarantees that any compact oriented Riemann surface (M^2, g) (i.e., M has Gauss curvature $k \equiv -1$) has a degree $g + 1$ meromorphic function $f: M \rightarrow S^2$ [9]. By uniformization theory any compact surface of genus g is conformally equivalent to such a Riemann surface; i.e., it is isometric to $(M^2, \varphi_1 g)$, where (M^2, g) is as above. The meromorphic map f satisfies the conditions of (0.4), with degree $D = (g + 1)$, so conclusion (0.5) follows.

(4.3) It is not clear exactly how the genus should effect a sharper estimate of μ_k (cf. Theorem (0.5)). One might expect that for low eigenvalues it could enter as it does in our result, but for higher eigenvalue estimates it should no longer be a factor. Indeed, M. Troyanov has recently shown such a result, albeit under certain curvature assumptions [11].

(4.4) We have not optimized the constant C in our Theorems (0.3)–(0.5) as much as possible. One could get a better constant by replacing η (in the intermediate region $\text{supp}(D\eta)$), with something which is closer to being harmonic with respect to the metric g_0 . Furthermore, the number “5” used repeatedly in §2 may be replaced with $(3 + 2\gamma)$ for any $\gamma > 0$, if the disjointness condition is modified from “doubly disjoint” to “ $(1 + \gamma)$ -disjoint.” (This will lead to new constants in the estimates.) Lastly, for low eigenvalue estimates one may proceed in a more ad-hoc manner, using these general decomposition ideas, to obtain much better bounds than the

general estimate for μ_k predicts. (For example, the author claims that using these ideas he can bound $\mu_2 = \lambda_1$ above for a conformal metric on S^2 , by $43\pi/A$. Of course, from the introduction we know that the sharp upper bound in $8\pi/A$.)

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Added in proof. It is an implicit assumption in all of our calculations that the variational approach to finding eigenvalues and eigenfunctions works for the domains or manifolds in question. In particular, our results only apply to the cases where the Rellich compactness theorem holds. For nondegenerate conformal metrics on compact manifolds or smooth compact subdomains this assumption is automatically satisfied, although for singular metrics or unbounded or singular subdomains it may not be so.

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